Section 1

Fourier Series

One-dimensional Fourier Series

f(x): Continuous and differential function; Periodic with period *L*.

Inverse Fourier transform (from $\hat{f}_k$ to *f*(x))

$$f(x) = \sum_{-\infty}^{\infty} \hat{f}_k \, exp(i\, Kx)$$

where the wavenumber *K=2πk/L* (k=integer).

kfactor = *2π/L.*

Forward transform (from *f*(x) to $\hat{f}_k$)

$$\hat{f}_k = \int_0^L f(x) \, exp(-i\, Kx)$$

Discrete Fourier Transform

Divide *L* into *N* parts:

$x_j = jL/N$ with $j=0:N-1$

$$f_j = \sum_{-N/2+1}^{N/2} \hat{f}_k \, exp(i\, 2\pi jk/N)$$

$$\hat{f}_k = \frac{1}{N} \sum_0^{N-1} f_j \, exp(-i\, 2\pi jk/N)$$

Linear transform between $\{f_j\}$ to $\{\hat{f}_k\}$.

Interestingly DFT is independent of *N*.

Properties

For real f(x)

$$\hat{f}_k = \hat{f}_{k+N}$$

$$\hat{f}_{-k} = \hat{f}_k^*$$

Energy (Parseval Theorem)

$$\frac{1}{L} \int dx (f(x))^2 = \sum_{-\infty}^{\infty} |\hat{f}_k|^2$$

For the discretized version, DFT defined earlier has the following property

$$\sum_0^{N-1} f_j^2 = \sum_{-N/2+1}^{N/2} |\hat{f}_k|^2$$

Mutlidimensional-dimensional Fourier Series

FFTW notation

$f(x)$ = periodic function in a box (L_1, L_2, ..,L_d) where d is the space dimension of the box

$f(\mathbf{x}) = \sum \hat{f}_k\ exp(\mathrm{i}\ \mathbf{K.x})$

where the wavenumber $K_s = 2\pi k_s/L_s$, (s=1:d) *with*

kfactor$_s$ = $2\pi/L_s$.

Forward transform (from f(x) to $\hat{f}_k$)

$\hat{f}_k = \int_0^L d\mathbf{x}\ \mathbf{f}(\mathbf{x})\ exp(-\mathrm{i}\ \mathbf{K.x})$

Discrete Multi-dimensional Transform

The vector **x** has component

$\mathbf{x} = \{ (L_1/N_1)\, j_1, (L_2/N_2) j_2, ..(L_2/N_2) j_2\}$

Inverse $f_{\mathbf{k}} = \sum f_{\mathbf{j}} exp(-i\frac{j_s k_s}{N_s})$

Forward $f_{\mathbf{j}} = \sum f_{\mathbf{k}} exp(i\frac{j_s k_s}{N_s})$

Properties

$\hat{f}(-\mathbf{k}) = \hat{f}^*(\mathbf{k})$

$\hat{f}(k_1+a_1 N_1, k_2+a_2 N_2, ...) = \hat{f}(k_1,k_2,..)$

About implementation

1. Compute Fourier transforms of the following real function:

(a) $f(x)=\cos(x) + 2 \sin(x)$

(b) $f(x,y) = 8 \cos(x) \sin(y)$

(c) $f(x,y) = \cos(x+y)$

(d) $f(x) = \cos^2(x)$

Assume periodic box of size 2Π in 1D, and of size (2Πx 2Π) for 2D.

What are the energies of the functions of problem 2? Verify Perceval's theorem. See my notes for the definition of energy and Perceval's theorem.

1. **Optional: Use fft (complex to complex) to compute Fourier transform of (a) $\exp(-i5x)$, (b) $\exp(i(x+y))$.**