Section 1

Classification of PDES

Linear PDEs are classified in three classes:

Parabolic

Elliptic

Hyperbolic

We will not delve into the properties of these types of equations. Rather, we will illustrate how we solve some of the PDEs encountered in physics.

Section 2

Solving diffusion equation using spectral method

Solve: $\partial_t \phi = \alpha \frac{\partial^2 \phi}{\partial x^2}$ with initial condition $\phi(x,0) = f(x)$, and vanishing boundary condition at x=0 and L.

It was first solved analytically by Fourier in 1807 using Fourier transform.

For a function ϕ(x) with a vanishing boundary condition in a box of length *L*:

$$\phi(x) = \sum_{k_n} \hat{\phi}(k_n)\sin(k_n x)$$

with $k_n = n\pi/L$

The boundary condition yields

$$\phi(0,t) = \phi(1,t) = 0.$$

First analyze the above equation in Fourier space:

$$\partial_t \hat{\phi}(k) = -\alpha k^2 \hat{\phi}(k) \quad (1)$$

which has an exact solution:

$$\hat{\phi}(k,t) = \hat{\phi}(k,t=0)\exp(-\alpha k^2 t)$$

Numerical solution:

To solve Eq. (1), we employ one of the simplest integration scheme--Euler's scheme that yields the following form of equation for every k:

$$\hat{\phi}^{(n+1)}(k) = \hat{\phi}^{(n)}(k)[1 - \alpha(\Delta t)k^2] \quad (2)$$

where (Δt) is the time step.

The stability criteria requires that

$|1 - \alpha(\Delta t)k^2| < 1$ for all k. Hence

´ $\alpha k_{\max}^2 \Delta t < 2.$

$\lambda/2 = h$ (grid spacing)

$$k_{\max} = \frac{2\pi}{\lambda} = \frac{\pi}{h}$$

Hence,

$$\Delta t < \frac{2h^2}{\pi^2 \alpha}$$

The aforementioned is called the spectral method since it involves Fourier modes.

Solving diffusion equation using finite difference scheme

Finite difference

Discretize into n segments.

$$\frac{d\phi_i(t)}{dt} = \frac{\alpha}{h^2}(\phi_{t+1} - 2\phi_i + \phi_{i-1})$$

Euler explicit

$$\phi_i^{(n+1)} = \phi_i^{(n)} + \frac{\alpha \Delta t}{h^2}(\phi_{i+1}^{(n)} - 2\phi_i^{(n)} + \phi_{i-1}^{(n)}) \qquad (1)$$

Inaccurate and unstable for large Δt.

Courant–Friedrichs–Lewy (CFL) condition

To test the stability of the above scheme, we attempt the following form of function:

$$\phi(x,t) = \exp(ikx)f(t) \qquad (2)$$

and substitute it in Eq. (1). Note that *k* correspond to one of the large-scale or small wavenumber Fourier mode. Also note that the solution of the diffusion equation decays to zero asymptotically. So, the numerical solution too should decay.

Substitution of Eq. (2) in Eq. (1) yields

$$\exp(ikx)f^{(n+1)} = \exp(ikx)[f^{(n)} + f^{(n)}\frac{2\alpha\Delta t}{h^2}(\cos(kh) - 1)]$$

Hence, the condition for stability is

$$|1 + \frac{2\alpha\Delta t}{h^2}(\cos(kh) - 1)| < 1.$$

Since $|\cos(kh) - 1| < 2$, the condition for stability is

$\frac{4\alpha\Delta t}{h^2} < 1$, or

$$\Delta t < \frac{h^2}{4\alpha}.$$

Compare the above condition with that obtain for the spectral method that yields $\Delta t < \frac{2h^2}{\pi^2\alpha}$. These two limits are approximately same (compare $\pi^2/2$ with 4).

Euler's scheme is inaccurate. For better accuracy, we can employ RK2 scheme that yields:

$$\phi_i^{(n+1/2)} = \phi_i^{(n)} + \frac{\alpha\Delta t}{h^2}(\phi_{i+1}^{(n)} - 2\phi_i^{(n)} + \phi_{i-1}^{(n)})$$

$$\phi_i^{(n+1)} = \phi_i^{(n)} + \frac{\alpha\Delta t}{h^2}(\phi_{i+1}^{(n+1/2)} - 2\phi_i^{(n+1/2)} + \phi_{i-1}^{(n+1/2)})$$

RK2 scheme up accurate up to $(\Delta t)^2$ per step.

The stability condition $\Delta t < \dfrac{h^2}{4\alpha}$ is too stringent for small *h,* that is it require too small Δt. There is a way to solve this problem, which is by employing a semi-implicit scheme called Crank Nickelson Scheme.

Crank Nickelson Scheme

For time stepping we employ

$$\phi_i^{(n+1)} = \phi_i^{(n)} + \frac{\alpha \Delta t}{2h^2}\Big[(\phi_{i+1}^{(n)} - 2\phi_i^{(n)} + \phi_{i-1}^{(n)})$$

$$+(\phi_{i+1}^{(n+1)} - 2\phi_i^{(n+1)} + \phi_{i-1}^{(n+1)})\Big]$$

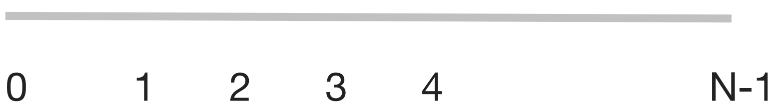
or

$$-\frac{\alpha\Delta t}{2h^2}\phi_{i-1}^{(n+1)} + \left(1 + \frac{\alpha\Delta t}{h^2}\right)\phi_i^{(n+1)} - \frac{\alpha\Delta t}{2h^2}\phi_{i+1}^{(n+1)}$$

$$= \phi_i^{(n)} + \frac{\alpha\Delta t}{2h^2}\left[\phi_{i+1}^{(n)} - 2\phi_i^{(n)} + \phi_{i-1}^{(n)}\right]$$

that yields a tridiagonal matrix.

Vanishing BC: Solve for $\phi_1, \ldots \phi_{n-2}$ (leave out ϕ_0, ϕ_{n-1} that are zeros).

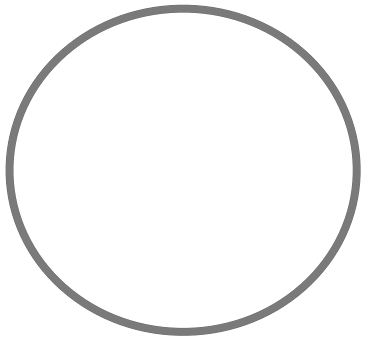


Matrix

$$\begin{pmatrix} X & Y & & & \\ Y & X & Y & & \\ & Y & X & Y & \\ & \ddots & \ddots & \ddots & \\ & & & Y & X \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_{N-2}\end{pmatrix} = \begin{pmatrix} r_1\\ r_2\\ r_3\\ r_4\\ r_{N-2}\end{pmatrix}$$

where $X = \left(1 + \dfrac{\alpha\Delta t}{h^2}\right)$ and $Y = -\dfrac{\alpha\Delta t}{2h^2}$.

For periodic BC:



Labels points on the circle as 0, 1, 2, .. N-1

$$\begin{pmatrix} X & Y & & & Y \\ Y & X & Y & & \\ & Y & X & Y & \\ & \ddots & \ddots & \ddots & \\ Y & & & Y & X \end{pmatrix}\begin{pmatrix}\phi_0\\ \phi_1\\ \phi_2\\ \phi_3\\ \phi_{N-1}\end{pmatrix} = \begin{pmatrix} r_0\\ r_1\\ r_2\\ r_3\\ r_{N-2}\end{pmatrix}$$

We solve for ϕ by solving the above tridiagonal matrix. The algorithm to solve tridiagonal matrices will be discussed along with other linear-algebra solvers.

Solving wave equation using finite difference scheme

Wave equation

$$\partial_t \phi + c\partial_x \phi = 0$$

with c>0. The wave moves along +x direction.

In Fourier space

$$\partial_t \hat{\phi}(k) = -ick\hat{\phi}(k) \qquad (1)$$

whose solution is $\hat{\phi}(k,t) = \hat{\phi}(k,0)\exp(ikct)$

General solution: $\phi(x,t) = \sum_k a_k \exp[ik(x-ct)]$

We solve the wave equation using finite difference scheme. Let's employ central-difference scheme for the space-derivative computation since it is accurate up to second order. For time stepping, we employ Euler's explicit scheme.

$$\phi_i^{(n+1)} = \phi_i^{(n)} - \frac{c\Delta t}{2h}(\phi_{i+1}^{(n)} - \phi_{i-1}^{(n)}) \qquad (2)$$

For stability, we attempt $\phi(x,t) = \exp(ikx)f(t)$ and substitute in Eq. (2) that yields

$$f^{(n+1)} = f^{(n)}[1 - ic\Delta t\frac{\sin kh}{h}].$$

Since $|1 - ic\Delta t\frac{\sin kh}{h}| > 1$, the integrating scheme is unstable, as discussed in the ODE chapter.

The stability issue is solved if we employ upwind scheme:

$$\phi_i^{(n+1)} = \phi_i^{(n)} - \frac{c\Delta t}{h}(\phi_i^{(n)} - \phi_{i-1}^{(n)})$$

Stability test with $\phi(x,t) = \exp(ikx)f(t)$ yields

$$f^{(n+1)} = f^{(n)}\left[1 - \frac{c\Delta t}{h}\{1 - \exp(-ikh)\}\right].$$

Hence, the integrating scheme is stable if

$$\left|1 - \frac{c\Delta t}{h}\{1 - \exp(-ikh)\}\right| > 1$$

that yields

or $$\frac{c\Delta t}{h}(1-\cos kh)\left\{\frac{c\Delta t}{h}-1\right\}<0.$$

Since $\cos kh < 1$, the condition for stability is

$$\frac{c\Delta t}{h}<1.$$

The upwind scheme is often employed for solving wave equation, and also equations that has a front propagation.

Exercises:

1. Solve the diffusion equation in one dimension:

$$\partial_t \phi = \kappa \nabla^2 \phi$$

Plot Φ(x) at different times.
Take κ=10 and initial condition as

What are your choices of Δt and Δx? Run you code till 10 diffusive time unit.

2. Ferziger Exercise 5.3

Section 5

Burger's equation

Burgers equation

$\partial_t u + u\partial_x u = \nu \partial_x^2 u$

Apply upwind scheme or RK2 for the nonlinear and central difference for the diffusion term.

When $u_i^{(n)} > 0$

$$u_i^{(n+1)} = u_i^{(n)} - \frac{1}{h} u_i^{(n)}(u_i^{(n)} - u_{i-1}^{(n)}) + \frac{\nu \Delta t}{h^2}(u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)})$$

When $u_i^{(n)} < 0$

$$u_i^{(n+1)} = u_i^{(n)} - \frac{1}{h} u_i^{(n)}(u_{i+1}^{(n)} - u_i^{(n)}) + \frac{\nu \Delta t}{h^2}(u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)})$$

Two time scales: $\frac{h^2}{\nu}$ and $\frac{h}{U_{\max}}$. To be safe, use min of the two. For the space discretization

Δx: $\quad h \approx \frac{\nu}{U_{\max}}$

Choose $\nu = 10^{-2}$, $L = 2\pi$, $u(x,0) = \sin x$

KPZ equation

$\partial_t h = \frac{1}{2}(\partial_x h)^2 + \nu \partial_x^2 h$

Similar as above

Note $u = -\partial_x h$ relates the two equations

Fluid Equation

Incompressible NS equation

$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u}$

$\nabla\cdot\mathbf{u} = 0$

Leads to

$\nabla^2 p = -\nabla\cdot\left[\mathbf{u}\cdot\nabla\mathbf{u}\right]$

Poisson's equation

Equations:

$\partial_t u_i = -u_j\partial_j u_i - \partial p - \nu\nabla^2 u_i$

Spectral method:

dt and h?

FD Poission solver...